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Combinatorial sums and finite differences

Michael Z. Spivey

Department of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA 98416-1043, USA

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Abstract

We present a new approach to evaluating combinatorial sums by using finite differences. Let $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ be sequences with the property that $\Delta b_k = a_k$ for $k \geqslant 0$. Let $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and let $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$. We derive expressions for g_n in terms of h_n and for h_n in terms of g_n. We then extend our approach to handle binomial sums of the form $\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, $\sum_k \binom{n}{2k} a_k$, and $\sum_k \binom{n}{2k+1} a_k$, as well as sums involving unsigned and signed Stirling numbers of the first kind, $\sum_{k=0}^{n} \left[{n \atop k}\right] a_k$ and $\sum_{k=0}^{n} s(n,k) a_k$. For each type of sum we illustrate our methods by deriving an expression for the power sum, with $a_k = k^m$, and the harmonic number sum, with $a_k = H_k = 1 + 1/2 + \cdots + 1/k$. Then we generalize our approach to a class of numbers satisfying a particular type of recurrence relation. This class includes the binomial coefficients and the unsigned Stirling numbers of the first kind.

Keywords: Binomial coefficient; Binomial sum; Combinatorial sum; Finite difference; Stirling number

0. Introduction

Let $\{a_k\}$ and $\{b_k\}$ be sequences with the property that a_k is the finite difference of b_k, that is, $a_k = \Delta b_k \triangleq b_{k+1} - b_k$, for $k \geqslant 0$. Let $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and let $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$. In this paper we derive expressions for g_n in terms of h_n and for h_n in terms of g_n. Our approach uses the recurrence relation for the binomial coefficients, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, to obtain an expression for g_n in terms of h_n and h_{n+1}. This expression itself defines a recurrence relation, which we then solve to obtain h_n in terms of g_n. The latter expression reduces the problem of determining the binomial sum of a sequence of numbers $\{b_k\}$ to the problem of determining the binomial sum of its finite difference sequence $\{a_k\}$. If the second sum is known or easy to obtain, this method can make derivations of binomial identities fairly simple.

We then modify our approach to derive formulas for h_n and g_n in terms of each other for the similar sums $\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, $\sum_k \binom{n}{2k} a_k$, and $\sum_k \binom{n}{2k+1} a_k$, as well as for sums involving unsigned and signed Stirling numbers of the first kind, $\sum_{k=0}^{n} \left[{n \atop k}\right] a_k$ and $\sum_{k=0}^{n} s(n,k) a_k$. We illustrate the use of our formulas for each of these

E-mail address: mspivey@ups.edu.

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types of sums by deriving identities for the power sum and the harmonic number sum; i.e., where $a_k = k^m$, m a nonnegative integer, and $a_k = H_k$, $H_n = \sum_{k=1}^{n} 1/k$, respectively. Many of these identities are, of course, known, and can be found in texts such as Benjamin and Quinn [1], Charalambides [3], Comtet [4], Gould [5], Graham et al. [6], and Riordan [12]. Moreover, they can be proved by a variety of methods, including Riordan arrays [16], generating functions [18], and the mechanical summation procedures in Petkovšek et al. [11].

The purpose of this article is not so much to prove these identities, though, as it is to illustrate our approach. For instance, our method provides a new way of looking at combinatorial sums from the perspective of finite differences, showing how several of these combinatorial sums relate to each other. A comparison of Theorems 4 and 5, for example, gives some insight into why alternating binomial sums often have simpler expressions than do their binomial sum counterparts. Second, while we do not claim that our approach is better than any of the current standard approaches, in some instances it can produce short proofs. Finally, unlike the mechanical summation procedures, we do not require the terms in the sum to be hypergeometric.

In Section 1 we derive our expression for g_n in terms of h_n for binomial sums. We then use this expression to give a quick derivation of the binomial theorem and to obtain some formulas involving binomial sums of Fibonacci numbers. In Section 2 we derive our expression for h_n in terms of g_n for binomial sums, and we use this expression to obtain the binomial power sum and the binomial harmonic number sum. In Sections 3, 4, and 5, we give expressions for g_n and h_n in terms of each other for alternating binomial sums, for binomial sums of the form $\sum_k \binom{n}{2k} a_k$ and $\sum_k \binom{n}{2k+1} a_k$, and for unsigned and signed Stirling numbers of the first kind, respectively. In each of these three sections we also derive the corresponding expressions for the power sum and the harmonic number sum. In Section 6 we generalize our expressions for g_n and h_n in terms of each other to a wide class of numbers $R(n,k)$ (including the binomial coefficients and unsigned Stirling numbers of the first kind) satisfying a certain type of two-term recurrence, and we illustrate the use of these formulas by deriving the power sum $\sum_{k=0}^{n} R(n,k)k^m$ for a particular subset of this class. In Section 7 we summarize our results and mention an open question. Throughout we take n to be an integer, and n is assumed to be nonnegative unless specified otherwise.

1. The binomial transform of the finite difference of a sequence

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence, and let $\binom{n}{k}$ be the usual binomial coefficient for integers n and k; i.e.,

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots(1)}, & k \geqslant 0, \\ 0, & k < 0. \end{cases}$$

Define the *binomial transform* of $\{a_k\}$ to be the sequence $\{g_n\}$ given by $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. Finally, define the *indicator function* $[S]$ to be 1 if the statement S is true and 0 otherwise.

(The definition of the binomial transform allows for negative indices with the sequence $\{g_n\}$; in the case where n is negative we have $g_n = 0$. Sometimes it turns out to be useful to have the binomial transform defined for all n, as some of our proofs are cleaner than they would be otherwise. We could also allow negative indices with the sequence $\{a_k\}$ by simply defining $a_k = 0$ for negative k, but for the input sequence this adds nothing. Also, there are at least three different definitions of the binomial transform in existence. The one used by Knuth [8, p. 136] is what we call the *alternating binomial transform* in Section 3. MathWorld's definition [17] is the inverse of our definition. Our definition is that used by the On-Line Encyclopedia of Integer Sequences [15].)

The following result uses the well-known recurrence relation for the binomial coefficients, $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$, valid for all integers n and k, to relate the binomial transform of the sequence $\{a_{k+1}\}$ to that of the sequence $\{a_k\}$.

Theorem 1. *For all integers n, if $\{g_n\}$ is the binomial transform of $\{a_k\}$, then $\{g_{n+1} - g_n - a_0[n=-1]\}$ is the binomial transform of $\{a_{k+1}\}$.*

Proof. If $n \geqslant 0$, we have

$$g_{n+1} - g_n = \sum_{k=0}^{n+1} \binom{n+1}{k} a_k - \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{k=0}^{n+1} \binom{n}{k-1} a_k = \sum_{k=0}^{n} \binom{n}{k} a_{k+1}.$$

(Since $\binom{n}{-1} = 0$, a_0 drops out of the sum.)

If $n = -1$, then $g_{n+1} - g_n - a_0[n = -1] = g_0 - g_{-1} - a_0 = a_0 - a_0 = 0 = \sum_{k=0}^{n} \binom{n}{k} a_{k+1}$. Finally, if $n \leqslant -2$, then $g_{n+1} - g_n = 0 = \sum_{k=0}^{n} \binom{n}{k} a_{k+1}$ as well. $\square$

Using Theorem 1, we can express the binomial transform g of $\{a_k\}$ in terms of the binomial transform h of $\{b_k\}$, like so:

Theorem 2. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for $k \geqslant 0$. If $\{g_n\}$ and $\{h_n\}$ are the binomial transforms of $\{a_k\}$ and $\{b_k\}$, respectively, then, for all n, $g_n = h_{n+1} - 2h_n - b_0[n = -1]$.*

Proof. If $n \geqslant 0$, we have

$$h_{n+1} - 2h_n = h_{n+1} - h_n - h_n = \sum_{k=0}^{n} \binom{n}{k} b_{k+1} - \sum_{k=0}^{n} \binom{n}{k} b_k = \sum_{k=0}^{n} \binom{n}{k} a_k = g_n.$$

In the case $n = -1$, we have $h_0 - 2h_{-1} - b_0 = b_0 - b_0 = 0 = g_{-1}$. For $n \leqslant -2$, g_n, h_{n+1}, and h_n are all 0. $\square$

Theorem 2 alone can be used to find binomial sums in which Δa_k and a_k share a close relationship. For example, it can be used to obtain a fairly quick proof of the binomial theorem for nonnegative integer values of n.

Identity 1. $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n$.

Proof. If $a_k = x^k$, then $\Delta a_k = (x - 1)x^k = (x - 1)a_k$. By Theorem 2, then, the binomial transform $\{g_n\}$ of $\{a_k\}$ satisfies the relation $(x - 1)g_n = g_{n+1} - 2g_n$ for $n \geqslant 0$. Thus $g_{n+1} = (x + 1)g_n$. Since $g_0 = 1$, the solution to this recurrence is $g_n = (x + 1)^n$, yielding $\sum_{k=0}^{n} \binom{n}{k} x^k = (x + 1)^n$. Replacing x in this equation with x/y and multiplying both sides by y^n completes the proof for $y \neq 0$. If $y = 0$, the identity reduces to $x^n = x^n$ (with $0^0 = 1$). $\square$

Theorem 2 can also be used to derive some known binomial sums involving the Fibonacci numbers $\{F_n\}$. While these sums may be obtained more easily via other methods, such as by Binet's formula [6, p. 299], we mention them here in order to illustrate the use of Theorem 2. By definition, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Identity 2. $\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$.

Identity 3. $\sum_{k=0}^{n} \binom{n}{k} F_{k+1} = F_{2n+1}$.

Proof. Let $a_k = F_k$, and let $b_k = F_{k+1}$. Let $g_n = \sum_{k=0}^{n} \binom{n}{k} F_k$, and let $h_n = \sum_{k=0}^{n} \binom{n}{k} F_{k+1}$. Since $F_k = F_{k+2} - F_{k+1}$, $\Delta b_k = a_k$. By Theorem 2, $h_{n+1} - 2h_n - F_1[n = -1] = g_n$ for all n. However, since the b_k sequence is just the a_k sequence shifted left once, Theorem 1 says that $g_{n+1} - g_n - F_0[n = -1] = h_n$. Thus we have the system of recurrences $g_{n+1} - g_n = h_n$ and $h_{n+1} - 2h_n = g_n + [n = -1]$. A little algebra shows that $g_n = 3g_{n-1} - g_{n-2} + [n = 1]$ and $h_n = 3h_{n-1} - h_{n-2} + [n = 0] - [n = 1]$. It is known that the Fibonacci sequence satisfies $F_n = F_3 F_{n-2} + F_2 F_{n-3}$ [9, p. 88]. Thus we have $F_n = 2F_{n-2} + F_{n-3}$, which implies, via $F_{n-3} = F_{n-2} - F_{n-4}$, that $F_n = 3F_{n-2} - F_{n-4}$. Since $g_0 = 0 = F_0$ and $g_1 = 1 = F_2$, the sequence $\{g_n\}$ has the same initial conditions and satisfies the same recurrence relation as the even Fibonacci numbers. Similarly, since $h_0 = 1 = F_1$ and $h_1 = 2 = F_3$, the sequence $\{h_n\}$ has the same initial conditions and satisfies the same recurrence relation as the odd Fibonacci numbers. Thus $g_n = F_{2n}$ and $h_n = F_{2n+1}$. $\square$

Define the *ordinary power series generating function*, or *opsgf*, of a sequence $\{a_k\}$ to be the formal power series $\sum_{k=0}^{\infty} a_k x^k$. (See Wilf [18] for an extensive discussion of generating functions and their uses.)

The following result places Identity 2 in a wider context.

Theorem 3. *The opsgf* $G(z)$ *of* $\sum_{k=0}^{n} \binom{n}{k} c^k F_k$ *is given by*

$$G(z) = \frac{cz}{1-(c+2)z-(c^2-c-1)z^2}.$$

Proof. Let $a_k = c^k F_k$, and let $b_k = c^{k+1} F_{k+1}$. Let $g_n = \sum_{k=0}^{n} \binom{n}{k} c^k F_k$, and let $h_n = \sum_{k=0}^{n} \binom{n}{k} c^{k+1} F_{k+1}$. We have $\Delta b_k = c^{k+2} F_{k+2} - c^{k+1} F_{k+1} = c^{k+1}(cF_{k+2} - F_{k+1}) = c^{k+1}((c-1)F_{k+1} + cF_k) = (c-1)b_k + c^2 a_k$. By Theorem 2, then, $h_{n+1} - 2h_n - c[n=-1] = (c-1)h_n + c^2 g_n$ for all integers n. Since the b_k sequence is just the a_k sequence shifted left once, Theorem 1 says that $g_{n+1} - g_n = h_n$ for all integers n. Solving the system of recurrences $h_{n+1} - 2h_n - c[n=-1] = (c-1)h_n + c^2 g_n$, $g_{n+1} - g_n = h_n$ for g_n, we obtain $g_n = (c+2)g_{n-1} + (c^2-c-1)g_{n-2} + c[n=1]$. Thus the opsgf $G(z)$ of g_n satisfies the equation $G(z) - z(c+2)G(z) - z^2(c^2-c-1)G(z) = cz$. (See [6, Chapter 7] for details.) Solving for $G(z)$ completes the proof. □

A few special cases of Theorem 3 produce particularly clean results.

Identity 4.

$$\sum_{k=0}^{n} \binom{n}{k} \Phi^k F_k = \Phi(\Phi+2)^{n-1}[n \geqslant 1] \quad \textit{where } \Phi = \frac{1+\sqrt{5}}{2}.$$

Identity 5.

$$\sum_{k=0}^{n} \binom{n}{k} \phi^k F_k = \phi(\phi+2)^{n-1}[n \geqslant 1] \quad \textit{where } \phi = \frac{1-\sqrt{5}}{2}.$$

Identity 6.

$$\sum_{k=0}^{n} \binom{n}{k} (-2)^k F_k = \begin{cases} (-2)5^{(n-1)/2} & \textit{if } n \textit{ is odd}, \\ 0 & \textit{if } n \textit{ is even}. \end{cases}$$

Identity 7.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = -F_n.$$

Proof. Take c to be Φ, ϕ, -2, and -1, respectively, in Theorem 3. In the first two cases, the expression $(c^2-c-1)z^2$ vanishes from the generating function, with the result that g is an exponential function of the form $c(c+2)^{n-1}$, for $n \geqslant 1$. In the third case, $(c+2)z$ vanishes from the generating function, resulting in g_n increasing by a factor of 5 with each two terms, starting with 0 when $n=0$ and -2 when $n=1$. In the last case, we have

$$G(z) = \frac{-z}{1-z-z^2},$$

which is the negative of the generating function for the Fibonacci numbers [6, p. 338]. □

2. The binomial transform of the antidifference of a sequence

Theorem 2 gives g_n in terms of h_n, where $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$, and $\Delta b_k = a_k$ for $k \geqslant 0$. This allows us to obtain results concerning sequences $\{a_k\}$ in which Δa_k and a_k are closely related. However, having h_n in terms of g_n would allow us to treat a much wider variety of sequences $\{a_k\}$ and $\{b_k\}$. The following theorem provides that relationship.

Theorem 4. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for each $k \geqslant 0$. If $\{g_n\}$ and $\{h_n\}$ are the binomial transforms of $\{a_k\}$ and $\{b_k\}$, respectively, then, for $n \geqslant 0$, $h_n = 2^n(b_0 + \sum_{k=1}^{n} \frac{g_{k-1}}{2^k})$.*

Proof. We know, from Theorem 2, that $h_{n+1} - 2h_n - b_0[n=-1] = g_n$. Thus we have $h_n - 2h_{n-1} = g_{n-1} + b_0[n=0]$. If $H(z)$ and $G(z)$ are the opsgf's for $\{h_n\}$ and $\{g_n\}$, respectively, this implies $H(z) - 2zH(z) = zG(z) + b_0$. Thus

$$H(z) = \frac{b_0 + zG(z)}{1-2z}.$$

Since $1/(1-2z)$ is the opsgf of the sequence $\{2^k\}$ [6, p. 335], and the product of the opsgf's of two sequences is the opsgf of the convolution of the two sequences [6, p. 333], this means that $\{h_n\}$ is the convolution of $\{2^k\}$ and the sequence given by

$$f_k = \begin{cases} b_0, & k=0, \\ g_{k-1}, & k \geqslant 1. \end{cases} \qquad \square$$

Theorem 4 allows us to produce several known binomial sum identities quite easily.

Identity 8. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. If $b_k = 1$ for each k, then $a_k = 0$ for each k. Thus $g_n = \sum_{k=0}^{n} \binom{n}{k} 0 = 0$. By Theorem 4, then, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\square$

Identity 9. $\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$.

Proof. If $b_k = k$ for each k, then $a_k = 1$ for each k. By Theorem 4 and Identity 8, then, we have $\sum_{k=0}^{n} \binom{n}{k} k = 2^n (0 + \sum_{k=1}^{n} \frac{2^{k-1}}{2^k}) = 2^n(\sum_{k=1}^{n} \frac{1}{2}) = \frac{n}{2} 2^n = n2^{n-1}$. $\square$

Identity 10. $\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1)2^{n-2}$.

Proof. If $b_k = k^2$ for each k, then $a_k = 2k+1$ for each k. By Theorem 4 and Identities 8 and 9, then, we have

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = 2^n \left(0+2\sum_{k=1}^{n}(k-1)\frac{2^{k-2}}{2^k} + \sum_{k=1}^{n} \frac{2^{k-1}}{2^k}\right) = 2^n \left(\sum_{k=1}^{n} k \frac{2^{k-1}}{2^k}\right) = 2^{n-1}\frac{n(n+1)}{2} = n(n+1)2^{n-2}. \quad \square$$

We could continue this process to obtain the binomial transforms of higher powers of k. However, it makes more sense to consider higher falling powers of k, as these have simpler finite differences, and then convert back to ordinary powers using the known relationship between the two kinds of powers. Let $k^{\underline{m}}$ denote the *falling factorial* $k(k-1)(k-2)\cdots(k-m+1)$. The following results are well-known (see, for example, [6, pp. 48–50]):

$$\Delta k^{\underline{m}} = mk^{\underline{m-1}}, \tag{1}$$

$$\sum_{k=1}^{n} k^{\underline{m}} = \frac{(n+1)^{\underline{m+1}}}{m+1}. \tag{2}$$

Identity 11. $\sum_{k=0}^{n} \binom{n}{k} k^{\underline{m}} = n^{\underline{m}} 2^{n-m}$.

Proof. By Identities 8 and 9, the statement is true for $m=0$ and 1. Fix $m\geqslant 2$, and assume that the statement is true for $1, 2, \ldots, m-1$. By (1), (2), and Theorem 4, then, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}k^{\underline{m}} &= 2^n\left(m\sum_{k=1}^{n}\frac{(k-1)^{\underline{m-1}}2^{k-1-(m-1)}}{2^k}\right)\\ &= 2^{n-m}\left(m\sum_{k=1}^{n}(k-1)^{\underline{m-1}}\right)\\ &= 2^{n-m}n^{\underline{m}}. \qquad \square\end{aligned}$$

Since k^m can be expressed as a sum of falling powers of k using the Stirling numbers of the second kind (sometimes known as Stirling subset numbers) $\left\{ {n \atop k} \right\}$, namely, $k^m=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}k^{\underline{j}}$ [6, p. 262], we have

Identity 12.

$$\sum_{k=0}^{n}\binom{n}{k}k^m=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n}{j}j!\,2^{n-j}.$$

Proof.

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}k^m &= \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n}{k}k^{\underline{j}}=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\sum_{k=0}^{n}\binom{n}{k}k^{\underline{j}}\\ &= \sum_{j=0}^{m}\left\{ {m \atop j} \right\}n^{\underline{j}}\,2^{n-j}=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n}{j}j!\,2^{n-j}.\end{aligned}$$

The last step follows from the fact that falling powers can be expressed as binomial coefficients: $n^{\underline{j}}=\binom{n}{j}j!$ [6, p. 154]. $\square$

Identity 12 is similar to a known expression for the power sum [14, p. 199]:

$$\sum_{k=0}^{n}k^m=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n+1}{j+1}j!.$$

We can also use Theorem 4 to find an expression for the binomial transform of the harmonic numbers. First, though, we need the following result.

Identity 13. $\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}=\frac{2^{n+1}-1}{n+1}$.

Proof.

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}=\frac{2^{n+1}-1}{n+1}.$$

(The first equality follows from the known identity $\binom{n}{k}\frac{n+1}{k+1}=\binom{n+1}{k+1}$ [6, p. 157].) $\square$

Let H_n denote the nth partial sum of the harmonic series; i.e., $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, with $H_0=0$. Identity 13 and Theorem 4 yield the following.

Identity 14. $\sum_{k=0}^{n}\binom{n}{k}H_k=2^n(H_n-\sum_{k=1}^{n}\frac{1}{k2^k})$.

Proof. We have $\Delta H_k = H_{k+1} - H_k = \frac{1}{k+1}$. Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2^n \left(\sum_{k=1}^{n} \frac{2^k - 1}{k\, 2^k} \right) = 2^n \left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k\, 2^k} \right) = 2^n \left(H_n - \sum_{k=1}^{n} \frac{1}{k\, 2^k} \right). \qquad \square$$

It is known that $\lim_{n\to\infty} \sum_{k=1}^{n} 1/(k\, 2^k) = \ln 2$ (substitute $-1/2$ into the Maclaurin series for $\ln(1+x)$), and thus we have the following expression for $\ln 2$:

$$\ln 2 = \lim_{n\to\infty} \left(H_n - 2^{-n} \sum_{k=0}^{n} \binom{n}{k} H_k \right).$$

3. Alternating binomial transforms

Define the *alternating binomial transform* of $\{a_k\}$ to be $\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$. (As noted in Section 1, Knuth [8, p. 136], defines this to be the binomial transform.) The version of Theorem 4 for the alternating binomial transform yields a much simpler relationship between g_n and h_n.

Theorem 5. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for each $k \geqslant 0$. If $\{g_n\}$ and $\{h_n\}$ are the alternating binomial transforms of $\{a_k\}$ and $\{b_k\}$, respectively, then, for all n, $g_n = -h_{n+1} + b_0[n=-1]$ and $h_n = -g_{n-1} + b_0[n=0]$.*

Proof. By Theorem 1, if $\{h_n^*\}$ is the binomial transform of $\{(-1)^k b_k\}$, then $\{h_{n+1}^* - h_n^* - b_0[n=-1]\}$ is the binomial transform of $\{(-1)^{k+1} b_{k+1}\}$. Thus $\{h_n^* - h_{n+1}^* + b_0[n=-1]\}$ is the binomial transform of $\{(-1)^k b_{k+1}\}$, and $\{h_n^* - h_{n+1}^* + b_0[n=-1] - h_n^*\} = \{-h_{n+1}^* + b_0[n=-1]\}$ is the binomial transform of $\{(-1)^k (b_{k+1} - b_k)\} = \{(-1)^k a_k\}$. Since $h_n = h_n^*$, we have $g_n = -h_{n+1} + b_0[n=-1]$. As this relationship holds for all n, $h_n = -g_{n-1} + b_0[n=0]$ as well. $\square$

As with Theorem 4, Theorem 5 can be used to determine a number of binomial sums quite easily.

Identity 15. $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = [n=0]$.

Proof. If $b_k = 1$ for each k, then $a_k = 0$ for each k. Since $g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, 0 = 0$, $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = [n=0]$. $\square$

Identity 16. $\sum_{k=0}^{n} \binom{n}{k} (-1)^k k = -[n=1]$.

Proof. If $b_k = k$ for each k, then $a_k = 1$ for each k. Since, by Identity 15, $g_n = [n=0]$, Theorem 5 yields $\sum_{k=0}^{n} \binom{n}{k} (-1)^k k = -[n=1]$. $\square$

Continuing in this vein, we obtain:

Identity 17. $\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{\underline{m}} = (-1)^m m! [n=m]$.

Proof. By the previous two results, the statement is true for $m=0$ and 1. Fix $m \geqslant 2$, and assume that the statement is true for $1, 2, \ldots, m-1$. Since $\Delta k^{\underline{m}} = m k^{\underline{m-1}}$, we have, by Theorem 5,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{\underline{m}} &= m(-1) \left((-1)^{m-1} (m-1)!\, [n-1 = m-1] \right) \\ &= (-1)^m m! [n=m]. \qquad \square \end{aligned}$$

As with Identity 12, this leads to the alternating binomial transform of $\{a_k\} = \{k^m\}$.

Identity 18. $\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = \left\{ {m \atop n} \right\} (-1)^n n!$.

Proof.

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^m = \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{{m \atop j}\right\}\binom{n}{k}(-1)^k k^{\underline{j}} = \sum_{j=0}^{m}\left\{{m \atop j}\right\}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^{\underline{j}}$$
$$= \sum_{j=0}^{m}\left\{{m \atop j}\right\}(-1)^j j![n=j] = \left\{{m \atop n}\right\}(-1)^n n!. \qquad \square$$

We can also use Theorem 5 to obtain the alternating binomial transform of the harmonic numbers. First, though, we need the following result.

Identity 19. $\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1} = \frac{1}{n+1}[n \geqslant 0]$.

Proof. Let $g_n = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}$. Then

$$-(n+1)g_n = -\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{n+1}{k+1} = \sum_{k=0}^{n}(-1)^{k+1}\frac{n+1}{k+1}\frac{n^{\underline{k}}}{k!}$$
$$= \sum_{k=0}^{n}(-1)^{k+1}\frac{(n+1)^{\underline{k+1}}}{(k+1)!} = \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{k+1} = [n+1=0]-1.$$

Dividing both sides of this equation by $-(n+1)$ completes the proof. $\square$

Identity 19 allows us to determine the alternating binomial transform of the harmonic numbers quite easily.

Identity 20. $\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n}[n \geqslant 1]$.

Proof. Since $\Delta H_k = \frac{1}{k+1}$, Theorem 5 and Identity 19 yield

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n}[n \geqslant 1]. \qquad \square$$

Many alternating binomial transforms have simpler expressions than the corresponding binomial transforms, as we can see from a comparison of the examples in this section and the previous one. The simplicity of Theorem 5 as compared to Theorem 4 reflects that as well.

4. Aerated binomial transforms

Theorem 4 can also be modified so that it applies to sums such as $\sum_k\binom{n}{2k}a_k$ and $\sum_k\binom{n}{2k+1}a_k$. The approach is to consider sums such as these as binomial transforms of *aerated sequences*, that is, sequences of the form $(a_0, 0, a_1, 0, a_2, 0, a_3, 0, \ldots)$ and $(0, a_0, 0, a_1, 0, a_2, 0, a_3, \ldots)$, respectively. We refer to $\sum_k\binom{n}{2k}a_k$ as the *even aerated binomial transform* of $\{a_k\}$ and $\sum_k\binom{n}{2k+1}a_k$ as the *odd aerated binomial transform* of $\{a_k\}$.

Theorem 6. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for each $k \geqslant 0$.*

1. *If $\{g_n\}$ and $\{h_n\}$ are the even aerated binomial transforms of $\{a_k\}$ and $\{b_k\}$, respectively, then, for all n,*

$$g_n = h_{n+2} - 2h_{n+1} + b_0[n=-1] - b_0[n=-2],$$

and, for $n \geqslant 0$,

$$h_n = 2^{n-1} b_0 + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} + \frac{b_0}{2}[n=0].$$

2. *If* $\{g_n\}$ *and* $\{h_n\}$ *are the odd aerated binomial transforms of* $\{a_k\}$ *and* $\{b_k\}$, *respectively, then, for all* n,

$$g_n = h_{n+2} - 2h_{n+1} - b_0[n=-1],$$

and, for $n \geqslant 0$,

$$h_n = 2^{n-1} b_0 + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} - \frac{b_0}{2}[n=0].$$

Proof. (Part 1) To shift the sequence $(b_0, 0, b_1, 0, b_2, 0, b_3, 0, \ldots)$ left twice, we apply Theorem 1 twice. Thus the binomial transform of $(b_1, 0, b_2, 0, b_3, 0, \ldots)$ is given by $\{h_{n+2} - h_{n+1} - b_0[n=-2] - (h_{n+1} - h_n - b_0[n=-1]) - 0[n=-1]\}$. Simplifying, this is $\{h_{n+2} - 2h_{n+1} + h_n + b_0[n=-1] - b_0[n=-2]\}$. Since $a_k = b_{k+1} - b_k$ for $k \geqslant 0$, for all n we have

$$\begin{aligned} g_n &= h_{n+2} - 2h_{n+1} + h_n + b_0[n=-1] - b_0[n=-2] - h_n \\ &= h_{n+2} - 2h_{n+1} + b_0[n=-1] - b_0[n=-2]. \end{aligned}$$

This expression defines a recurrence relation $h_n - 2h_{n-1} = g_{n-2} - b_0[n=1] + b_0[n=0]$, valid for all n. If $H(z)$ and $G(z)$ are the opsgf's for $\{h_n\}$ and $\{g_n\}$, respectively, then we have $H(z) - 2zH(z) = z^2 G(z) - b_0 z + b_0$, or

$$H(z) = \frac{z^2 G(z) - b_0 z + b_0}{1 - 2z}.$$

Therefore, $\{h_n\}$ is the convolution of $\{2^k\}$ and the sequence given by

$$f_k = \begin{cases} b_0 & \text{if } k = 0, \\ -b_0 & \text{if } k = 1, \\ g_{k-2} & \text{if } k \geqslant 2. \end{cases}$$

Thus, for $n \geqslant 0$,

$$h_n = 2^n b_0 - 2^{n-1} b_0 [n \geqslant 1] + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} = 2^{n-1} b_0 + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} + \frac{b_0}{2}[n=0].$$

(Part 2) The proof in this case is similar to that in Part 1. The binomial transform of $(0, b_1, 0, b_2, 0, b_3, \ldots)$ is $\{h_{n+2} - h_{n+1} - 0[n=-2] - (h_{n+1} - h_n - 0[n=-1]) - b_0[n=-1]\}$. This yields the recurrence $h_n - 2h_{n-1} = g_{n-2} + b_0[n=1]$, valid for all n. Thus $\{h_n\}$ is the convolution of $\{2^k\}$ and the sequence given by

$$f_k = \begin{cases} 0 & \text{if } k = 0, \\ b_0 & \text{if } k = 1, \\ g_{k-2} & \text{if } k \geqslant 2. \end{cases}$$

Therefore, for $n \geqslant 0$,

$$h_n = 2^{n-1} b_0 [n \geqslant 1] + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} = 2^{n-1} b_0 + 2^n \sum_{k=2}^{n} \frac{g_{k-2}}{2^k} - \frac{b_0}{2}[n=0]. \qquad \square$$

Identity 21. $\sum_k \binom{n}{2k} = 2^{n-1} + \frac{1}{2}[n=0]$.

Identity 22. $\sum_k \binom{n}{2k+1} = 2^{n-1} - \frac{1}{2}[n=0]$.

Proof. If $b_k=1$ for each k, then $a_k=0$ for each k. We then have $g_n=0$. By Theorem 6, then, $\sum_k \binom{n}{2k}=2^{n-1}+\frac{1}{2}[n=0]$ and $\sum_k \binom{n}{2k+1}=2^{n-1}-\frac{1}{2}[n=0]$. $\square$

Identity 23. $\sum_k \binom{n}{2k} k = n2^{n-3}[n\geqslant 2]$.

Identity 24. $\sum_k \binom{n}{2k+1} k = (n-2)2^{n-3}[n\geqslant 2]$.

Proof. If $b_k=k$ for each k, then $a_k=1$ for each k. By Theorem 6 and Identity 21, then,

$$\begin{aligned}\sum_k \binom{n}{2k} k &= 2^n \sum_{k=2}^{n} \frac{2^{k-3}+\frac{1}{2}[k=2]}{2^k} = 2^n \sum_{k=2}^{n} \frac{1}{8} + \frac{2^n}{8}[n\geqslant 2]\\ &= \left((n-1)2^{n-3}+2^{n-3}\right)[n\geqslant 2] = n2^{n-3}[n\geqslant 2].\end{aligned}$$

The proof of Identity 24 is similar. $\square$

Identity 25. *If* $m\geqslant 1$, *then* $\sum_k \binom{n}{2k} k^{\underline{m}} = n(n-m-1)^{\underline{m-1}}\,2^{n-2m-1}[n\geqslant m+1]$.

Identity 26. *If* $m\geqslant 1$, *then* $\sum_k \binom{n}{2k+1} k^{\underline{m}} = (n-m-1)^{\underline{m}}\,2^{n-2m-1}[n\geqslant m+1]$.

Proof. By Identity 23, Identity 25 is true in the case $m=1$. Fix $m\geqslant 2$, and suppose that Identity 25 is true for $1,2,\ldots,m-1$. By Theorem 6, we have

$$\begin{aligned}\sum_k \binom{n}{2k} k^{\underline{m}} &= 2^n \sum_{k=2}^{n} \frac{m2^{k-2-2m+1}(k-2)(k-2-m+1-1)^{\underline{m-2}}[k\geqslant m+2]}{2^k}\\ &= 2^{n-2m-1} \sum_{k=m+2}^{n} m(k-2)(k-m-2)^{\underline{m-2}}.\end{aligned}$$

Applying summation by parts [6, p. 55], this expression becomes

$$\begin{aligned}&2^{n-2m-1}\left(\frac{m(k-2)(k-m-2)^{\underline{m-1}}}{m-1}\Bigg|_{k=m+2}^{n+1} - \sum_{k=m+2}^{n} \frac{m(k-m-1)^{\underline{m-1}}}{m-1}\right)[n\geqslant m+2]\\ &= 2^{n-2m-1}\left[\frac{m(k-2)(k-m-2)^{\underline{m-1}}}{m-1} - \frac{(k-m-1)^{\underline{m}}}{m-1}\right]_{k=m+2}^{n+1}[n\geqslant m+2]\\ &= 2^{n-2m-1}\left(\frac{m(n-1)(n-m-1)^{\underline{m-1}}}{m-1} - \frac{(n-m)^{\underline{m}}}{m-1}\right)[n\geqslant m+2]\\ &= \frac{2^{n-2m-1}(n-m-1)^{\underline{m-1}}}{m-1}(mn-m-n+m)\,[n\geqslant m+2]\\ &= n(n-m-1)^{\underline{m-1}}\,2^{n-2m-1}[n\geqslant m+1].\end{aligned}$$

The proof for Identity 26 is similar (and is, in fact, easier, as summation by parts is not needed). $\square$

As with the binomial and alternating binomial transforms, we can use our result for falling powers to obtain the even and odd aerated binomial transform power sums.

Identity 27. *If* $m\geqslant 1$, $\sum_k \binom{n}{2k} k^m = n\sum_{j=1}^{\min\{m,n-1\}} \left\{ {m \atop j} \right\} \binom{n-j-1}{j-1} (j-1)!\,2^{n-2j-1}$.

Identity 28. If $m\geqslant 1$, $\sum_k \binom{n}{2k+1} k^m = \sum_{j=1}^{\min\{m,n-1\}} \left\{ {m \atop j} \right\} \binom{n-j-1}{j} j!\,2^{n-2j-1}$.

Proof. For the even aerated power sum, we have

$$\begin{aligned}
\sum_k \binom{n}{2k} k^m &= \sum_k \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \binom{n}{2k} k^{\underline{j}} \\
&= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_k \binom{n}{2k} k^{\underline{j}} \\
&= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} n(n-j-1)^{\underline{j-1}} 2^{n-2j-1} [n \geqslant j+1] \\
&= \sum_{j=1}^{\min\{m,n-1\}} \left\{ {m \atop j} \right\} n(n-j-1)^{\underline{j-1}} 2^{n-2j-1} \\
&= n \sum_{j=1}^{\min\{m,n-1\}} \left\{ {m \atop j} \right\} \binom{n-j-1}{j-1} (j-1)!\, 2^{n-2j-1}.
\end{aligned}$$

The proof of Identity 28 is similar. $\square$

We can also consider aerated binomial transforms of the harmonic numbers. First, though, we need the following identities.

Identity 29. $\sum_k \binom{n}{2k} \frac{1}{k+1} = \frac{n2^{n+1}+2}{(n+1)(n+2)}$.

Identity 30. $\sum_k \binom{n}{2k+1} \frac{1}{k+1} = \frac{2^{n+1}-2}{n+1}$.

Proof. Let $g_n = \sum_k \binom{n}{2k} \frac{1}{k+1}$. Letting $\sum g_n$ denote the antidifference of g_n, we have

$$\begin{aligned}
&\frac{1}{2} g_n = \frac{1}{2} \sum_k \binom{n}{2k} \frac{1}{k+1} = \sum_k \frac{n^{\underline{2k}}}{2(k+1)(2k)!} \\
&\Rightarrow \frac{1}{2} \sum g_n = C + \sum_k \frac{n^{\underline{2k+1}}}{(2k+2)(2k+1)(2k)!} \\
&\Rightarrow (n+1)\frac{1}{2} \sum g_n = C(n+1) + \sum_k \frac{(n+1)^{\underline{2k+2}}}{(2k+2)!} = C(n+1) + \sum_k \binom{n+1}{2k+2} = C(n+1) + 2^n - 1 \\
&\Rightarrow \sum g_n = 2C + \frac{2^{n+1}-2}{n+1}.
\end{aligned}$$

Taking the finite difference of $\sum g_n$, we obtain

$$\begin{aligned}
g_n &= 2C + \frac{2^{n+2}-2}{n+2} - 2C - \frac{2^{n+1}-2}{n+1} \\
&= \frac{(n+1)(2^{n+2}-2) - (n+2)(2^{n+1}-2)}{(n+1)(n+2)} \\
&= \frac{n2^{n+1}+2}{(n+1)(n+2)}.
\end{aligned}$$

The proof of Identity 30 is similar (but easier, as indefinite summation is not required). $\square$

We can now determine the aerated binomial transforms of the harmonic numbers.

Identity 31.

$$\sum_k \binom{n}{2k} H_k = 2^{n-1}\left(H_n - \sum_{k=1}^{n} \frac{1}{k2^{k-1}} + \frac{1}{n} - \frac{1}{n2^{n-1}}\right)[n \geqslant 1].$$

Identity 32.

$$\sum_k \binom{n}{2k+1} H_k = 2^{n-1}\left(H_n - \sum_{k=1}^{n} \frac{1}{k2^{k-1}} - \frac{1}{n} + \frac{1}{n2^{n-1}}\right)[n \geqslant 1].$$

Proof. By Theorem 6 and Identity 29, we have

$$\begin{aligned}
\sum_k \binom{n}{2k} H_k &= 2^n \sum_{k=2}^{n} \frac{(k-2)2^{k-1}+2}{2^k k(k-1)} \\
&= 2^n \left(\sum_{k=2}^{n} \frac{1}{2(k-1)} - \sum_{k=2}^{n} \frac{1}{k(k-1)} + \sum_{k=2}^{n} \frac{1}{2^{k-1}k(k-1)}\right) \\
&= 2^n \left(\frac{H_{n-1}}{2} + \frac{1}{n} - 1 - \left.\frac{1}{2^{k-1}(k-1)}\right|_{k=2}^{n+1} - \sum_{k=2}^{n} \frac{1}{k2^k}\right)[n \geqslant 2] \\
&= 2^n \left(\frac{H_{n-1}}{2} + \frac{1}{n} - 1 - \frac{1}{n2^n} + \frac{1}{2} - \sum_{k=1}^{n} \frac{1}{k2^k} + \frac{1}{2}\right)[n \geqslant 2] \\
&= 2^{n-1}\left(H_n - \sum_{k=1}^{n} \frac{1}{k2^{k-1}} + \frac{1}{n} - \frac{1}{n2^{n-1}}\right)[n \geqslant 1].
\end{aligned}$$

(We can replace $[n \geqslant 2]$ with $[n \geqslant 1]$ at the end because the expression equals 0 when $n = 1$, as does $\sum_k \binom{1}{2k} H_k$.) The proof of Identity 32 is similar. (Once again, though, it is easier, as indefinite summation and summation by parts are not required.) □

This technique can be extended, of course, for binomial sums of the form $\sum_k \binom{n}{pk+r} a_k$ for fixed p and r.

5. Sums involving Stirling numbers of the first kind

The approach to evaluating sums in the previous sections works because of the recurrence relation between the binomial coefficients: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. Other well-known combinatorial numbers satisfy simple recurrence relationships as well, so one might hope that the right modification of our technique would work on sums involving other combinatorial numbers. This turns out to be the case for Stirling numbers of the first kind, signed or unsigned. We denote the signed Stirling numbers of the first kind by $s(n,k)$. The unsigned Stirling numbers of the first kind are also known as the Stirling cycle numbers and are denoted $\left[{n \atop k}\right]$, with $s(n,k) = \left[{n \atop k}\right](-1)^{n-k}$. We consider the unsigned Stirling numbers of the first kind first. These satisfy the relationship $\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]$ [6, p. 261].

Theorem 7. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for $k \geqslant 0$. If $g_n = \sum_{k=0}^{n} \left[{n \atop k}\right] a_k$ and $h_n = \sum_{k=0}^{n} \left[{n \atop k}\right] b_k$, then, for all n,*

$$g_n = h_{n+1} - (n+1)h_n - b_0[n = -1]. \tag{3}$$

and, for $n \geqslant 0$,

$$h_n = n!\left(b_0 + \sum_{k=1}^{n} \frac{g_{k-1}}{k!}\right). \tag{4}$$

Proof. Since, for $n \geqslant 0$, $\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]$, we have

$$\begin{aligned} h_{n+1} - nh_n &= \begin{bmatrix} n+1 \\ 0 \end{bmatrix} b_0 + \begin{bmatrix} n+1 \\ 1 \end{bmatrix} b_1 + \begin{bmatrix} n+1 \\ 2 \end{bmatrix} b_2 + \cdots + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix} b_{n+1} \\ &\quad - n\left(\begin{bmatrix} n \\ 0 \end{bmatrix} b_0 + \begin{bmatrix} n \\ 1 \end{bmatrix} b_1 + \cdots + \begin{bmatrix} n \\ n \end{bmatrix} b_n\right) \\ &= \begin{bmatrix} n \\ 0 \end{bmatrix} b_1 + \begin{bmatrix} n \\ 1 \end{bmatrix} b_2 + \cdots + \begin{bmatrix} n \\ n \end{bmatrix} b_{n+1}. \end{aligned}$$

Thus, if $n \geqslant 0$, $h_{n+1} - (n+1)h_n = h_{n+1} - nh_n - h_n = \left[{n \atop 0}\right] a_0 + \left[{n \atop 1}\right] a_1 + \cdots + \left[{n \atop n}\right] a_n = g_n$.

If $n = -1$, then $h_0 - 0 - b_0 = b_0 - b_0 = 0 = g_{-1}$. If $n \leqslant -2$, h_{n+1}, h_n, and g_n are all 0. This completes the proof of (3).

Eq. (3) defines a recurrence relation $h_{n+1} - (n+1)h_n = g_n + b_0[n = -1]$. This recurrence relation does not have constant coefficients, and thus its solution is more complicated than those we have seen thus far. Techniques for solving such recurrence relations are known, however (see [10, pp. 46–48], for example), and we have

$$\begin{aligned} h_n &= \left(\prod_{k=0}^{n-1}(k+1)\right) b_0 + \left(\prod_{k=0}^{n-1}(k+1)\right) \sum_{k=0}^{n-1} \frac{g_k}{\prod_{i=0}^{k}(i+1)} \\ &= n!\left(b_0 + \sum_{k=1}^{n} \frac{g_{k-1}}{k!}\right). \qquad \square \end{aligned}$$

Identity 33. $\sum_{k=0}^{n} \left[{n \atop k}\right] = n! = \left[{n+1 \atop 1}\right]$.

Proof. If $b_k = 1$ for each $k \geqslant 0$, then $a_k = 0$ for each k. Clearly, $\sum_{k=0}^{n} \left[{n \atop k}\right] 0 = 0$. By Theorem 7, then, we have $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$, which is known to be equal to $\left[{n+1 \atop 1}\right]$ [6, p. 260]. $\square$

For the next couple of identities we need the following result about unsigned Stirling numbers of the first kind [6, p. 265].

Identity 34. $n!\sum_{k=0}^{n} \left[{k \atop m}\right]/k! = \left[{n+1 \atop m+1}\right]$.

Then we have:

Identity 35. $\sum_{k=0}^{n} \left[{n \atop k}\right] k = \left[{n+1 \atop 2}\right]$.

Proof. If $b_k = k$ for each $k \geqslant 0$, then $a_k = 1$ for each k. By Identities 33 and 34 and Theorem 7, then, we have $\sum_{k=0}^{n} \left[{n \atop k}\right] k = n!\sum_{k=1}^{n} \left[{k \atop 1}\right]/k! = \left[{n+1 \atop 2}\right]$. $\square$

Identity 36. $\sum_{k=0}^{n} \left[{n \atop k}\right] k^{\underline{m}} = \left[{n+1 \atop m+1}\right] m!$.

Proof. By Identities 33 and 35, this statement is true for $m = 0$ and 1. Fix $m \geqslant 2$, and suppose the statement to be true for values from 0 to $m-1$. By Identity 34 and Theorem 7, then, we have $\sum_{k=0}^{n} \left[{n \atop k}\right] k^{\underline{m}} = m\, n!\sum_{k=1}^{n} \left[{k \atop m}\right] (m-1)!/k! = \left[{n+1 \atop m+1}\right] m!$. $\square$

Using Stirling numbers of the second kind to convert from ordinary powers to falling powers, we obtain:

Identity 37. $\sum_{k=0}^{n} \left[{n \atop k}\right] k^m = \sum_{j=0}^{m} \left[{n+1 \atop j+1}\right] \left\{{m \atop j}\right\} j!$.

We can also derive an expression for the sum of the products of unsigned Stirling numbers of the first kind and harmonic numbers. First, though, we need a couple of definitions and a result. For $n \geqslant 0$, let $c_n = \int_0^1 x^{\bar{n}}\, \mathrm{d}x$, where $x^{\bar{n}}$ is known as a *rising power* and is given by $x^{\bar{n}} = x(x+1)(x+2)\cdots(x+n-1)$. The c_n's are called *Cauchy numbers of the second type* [4, p. 294] and are used in numerical integration. (When Newton's backwards difference formula is used to approximate $f(x)$, and the result is used to approximate $\int_0^1 f(x)\,\mathrm{d}x$, the coefficient of $\nabla^n f(0)$ in the approximation is $c_n/n!$ [2, pp. 433, 455].)

Identity 38. $\sum_{k=0}^{n} \left[{n \atop k}\right]/(k+1) = c_n$.

Proof. Using the fact that unsigned Stirling numbers of the first kind convert rising powers to ordinary powers [6, p. 263]; i.e, $x^{\bar{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$, we have $c_n = \int_0^1 x^{\bar{n}}\,\mathrm{d}x = \int_0^1 \sum_{k=0}^{n} \left[{n \atop k}\right] x^k\,\mathrm{d}x = \sum_{k=0}^{n} \left[{n \atop k}\right]/(k+1)$. $\square$

The following is then immediate from Theorem 7.

Identity 39. $\sum_{k=0}^{n} \left[{n \atop k}\right] H_k = n! \sum_{k=1}^{n} \frac{c_{k-1}}{k!}$.

All of these results may be repeated for the signed Stirling numbers of the first kind $s(n,k)$. The recursion for the $s(n,k)$ numbers is as follows: $s(n+1,k) = s(n,k-1) - ns(n,k)$ [3, p. 293].

Theorem 8. *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$ for $k \geqslant 0$. If $g_n = \sum_{k=0}^{n} s(n,k) a_k$ and $h_n = \sum_{k=0}^{n} s(n,k) b_k$, then, for all n,*

$$g_n = h_{n+1} + (n-1)h_n - b_0[n=-1] \tag{5}$$

and

$$h_n = (-1)^n (n-2)! \sum_{k=1}^{n-1} \frac{(-1)^{k-1} g_k}{(k-1)!} + b_0[n=0] + b_1[n=1]. \tag{6}$$

Proof. By the recursion for the $s(n,k)$ numbers, we can see that, for $n \geqslant 0$, $h_{n+1} + nh_n = s(n,0)b_1 + s(n,1)b_2 + \cdots + s(n,n)b_{n+1}$. Thus $h_{n+1} + (n-1)h_n = s(n,0)(b_1 - b_0) + s(n,1)(b_2 - b_1) + \cdots + s(n,n)(b_{n+1} - b_n) = g_n$. We have $h_0 = b_0$, and thus the case $n = -1$ of (5) is satisfied. If $n \leqslant -2$, all quantities in (5) are 0.

Eq. (5) yields the recurrence $h_{n+1} + (n-1)h_n = g_n + b_0[n=-1]$. Its solution is given by the following (again, see, for example, Mickens [10, pp. 46–48]).

$$\begin{aligned} h_n &= \left(\prod_{k=2}^{n-1} (-(k-1)) \right) \sum_{k=1}^{n-1} \frac{g_k}{\prod_{i=2}^{k} (-(i-1))} + b_0[n=0] + (b_0 + g_0)[n=1] \\ &= (-1)^n (n-2)! \sum_{k=1}^{n-1} \frac{(-1)^{k-1} g_k}{(k-1)!} + b_0[n=0] + b_1[n=1]. \qquad \square \end{aligned}$$

As with its analogues, Theorem 8 can be used to generate identities involving sums of Stirling numbers of the first kind.

Identity 40. $\sum_{k=0}^{n} s(n,k) = [n=0] + [n=1] = s(n-1,0) + s(n-1,-1)$.

Proof. If $b_k = 1$ for each k, then $a_k = 0$ for each k. Then, since $g_n = 0$ for each n, we have $\sum_{k=0}^{n} s(n,k) = [n=0] + [n=1]$. Using the conventions that $s(n,0) = [n=0]$ and $s[0,k] = [k=0]$ [6, p. 266], and extending the recurrence for the Stirling numbers of the first kind to allow for negative indices, we see that $s(-1,-1) = 1$. Thus $[n=0] + [n=1] = s(n-1,0) + s(n-1,-1)$. $\square$

Identity 41. $\sum_{k=0}^{n} s(n,k)k = [n=1] + (-1)^n (n-2)! [n \geqslant 2] = s(n-1,1) + s(n-1,0)$.

Proof. If $b_k = k$ for each k, then $a_k = 1$ for each k. By the previous result and Theorem 8, we have

$$\begin{aligned}\sum_{k=0}^{n} s(n,k)k &= (-1)^n(n-2)!\sum_{k=1}^{n-1}\frac{(-1)^{k-1}([k=0]+[k=1])}{(k-1)!} + [n=1]\\ &= (-1)^n(n-2)![n\geqslant 2] + [n=1]\\ &= s(n-1,1) + s(n-1,0).\end{aligned}$$

The last statement follows because $s(n,1) = (-1)^{n-1}(n-1)![n\geqslant 1]$. □

For the next two results, we need the corresponding version of Identity 34 for Stirling numbers of the first kind. (See, for example, [3, p. 295].)

Identity 42. $n!\sum_{k=0}^{n} s(k,m)(-1)^{n-k}/k! = s(n+1,m+1)$.

Then we have:

Identity 43. $\sum_{k=0}^{n} s(n,k)k^{\underline{m}} = m!(s(n-1,m) + s(n-1,m-1))$.

Proof. Identities 40 and 41 show that the statement is true for $m=0$ and 1. Fix $m\geqslant 2$, and assume that the statement is true for $0, 1, 2, \ldots, m-1$. By Theorem 8 and Identity 42, we have

$$\begin{aligned}\sum_{k=0}^{n} s(n,k)k^{\underline{m}} &= m(-1)^n(n-2)!\sum_{k=0}^{n-2}\frac{(-1)^k(m-1)!(s(k,m-1)+s(k,m-2))}{k!}\\ &= m!(n-2)!\sum_{k=0}^{n-2}\frac{(-1)^{n-k}(s(k,m-1)+s(k,m-2))}{k!}\\ &= m!(s(n-1,m)+s(n-1,m-1)). \qquad \square\end{aligned}$$

Once again, using Stirling numbers of the second kind to express ordinary powers in terms of falling powers, we obtain an expression for the power sum.

Identity 44. $\sum_{k=0}^{n} s(n,k)k^m = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} (s(n-1,j) + s(n-1,j-1))j!$.

We can use Theorem 8 to obtain an identity involving the sum of the product of harmonic numbers and Stirling numbers of the first kind. For the purposes of the rest of this section, let $b_n = \int_0^1 x^{\underline{n}}\,\mathrm{d}x$ for $n\geqslant 0$. The b_n's are known as *Cauchy numbers of the first type* [4, p. 294] or as *Bernoulli numbers of the second kind* [13, p. 114]. They are used in numerical integration; $b_n/n!$ is the coefficient of $\Delta^n f(0)$ in Gregory's formula for $\int_0^1 f(x)\,\mathrm{d}x$. (This is the same as if Newton's forward difference formula is first used to approximate $f(x)$. See [7, pp. 524–525], although here b_n is defined slightly differently, and [2, pp. 433, 455].)

Identity 45. $\sum_{k=0}^{n} \frac{s(n,k)}{k+1} = b_n$.

Proof. Since Stirling numbers of the first kind are used to convert from falling powers to ordinary powers [6, p. 263], we have $b_n = \int_0^1 x^{\underline{n}}\,\mathrm{d}x = \int_0^1 \sum_{k=0}^{n} s(n,k)x^k\,\mathrm{d}x = \sum_{k=0}^{n}\frac{s(n,k)}{k+1}$. □

The following result is then immediate upon applying Theorem 8.

Identity 46. $\sum_{k=0}^{n} s(n,k)H_k = (-1)^n(n-2)!\sum_{k=1}^{n-1}\frac{(-1)^{k-1}b_k}{(k-1)!} + [n=1]$.

Unfortunately, our approach for finding a relationship between h_n and g_n does not appear to work with sums such as $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} a_k$ involving Stirling numbers of the second kind. The recurrence relation for the Stirling numbers of the second kind, $\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$ [6, p. 259], contains an additional dependence on k that the Stirling numbers of the first kind do not have. An attempt to produce a recurrence for h_n and g_n as in Theorem 7 would fail at the point in the derivation in which we multiply h_n by n. This is because in the case of Stirling numbers of the second kind each term of h_n would need to be multiplied by a different value; in particular, term k would need to be multiplied by k. The same problem occurs with the Eulerian numbers $\left\langle {n \atop k} \right\rangle$, as their basic recurrence relation also has an extra dependence on k.

However, since even the simple sum $\sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ does not have a formula nearly as nice as the corresponding sums we have seen thus far, perhaps the failure of our approach on sums involving Stirling numbers of the second kind is to be expected. (The sum $\sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ is known as the nth *Bell number* [14, p. 118].) On the other hand, $\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle$ does have a nice form, $n!$.

6. Sums involving two-term recurrences

The relationships between g_n and h_n in the binomial transform case (Theorem 4) and with sums involving unsigned Stirling numbers of the first kind (Theorem 7) are similar enough to be striking:

$$h_n = 2^n \left(b_0 + \sum_{k=1}^{n} \frac{g_{k-1}}{2^k} \right)$$

and

$$h_n = n! \left(b_0 + \sum_{k=1}^{n} \frac{g_{k-1}}{k!} \right).$$

In fact, they are instances of a general relationship involving certain kinds of two-term recurrences.

Theorem 9. *Let $\{a_k\}$ and $\{b_k\}$ be sequences satisfying $\Delta b_k = a_k$ for $k \geqslant 0$. Let $f(n)$ be a function such that $f(n) \neq -1$ for any n. Let $\{R(n,k)\}$ be a set of numbers satisfying a two-term recurrence of the form $R(n+1,k) = f(n)R(n,k) + R(n,k-1)$ and boundary conditions $R(n,-1) = 0$ and $R(n,n+1) = 0$ when $n \geqslant 0$. Let $g_n = \sum_{k=0}^{n} R(n,k)a_k$, and let $h_n = \sum_{k=0}^{n} R(n,k)b_k$. Then*

$$g_n = h_{n+1} - (f(n)+1)h_n - R(0,0)b_0[n=-1], \tag{7}$$

and, for $n \geqslant 0$,

$$h_n = \left(\prod_{k=0}^{n-1} (f(k)+1) \right) \left(R(0,0)b_0 + \sum_{k=0}^{n-1} \frac{g_k}{\prod_{i=0}^{k} (f(k)+1)} \right). \tag{8}$$

Proof. If $n \geqslant 0$, we have

$$\begin{aligned} \sum_{k=0}^{n+1} R(n,k-1)b_k &= \sum_{k=0}^{n+1} (R(n+1,k) - f(n)R(n,k))b_k \\ &= \sum_{k=0}^{n+1} R(n+1,k)b_k - f(n) \sum_{k=0}^{n+1} R(n,k)b_k. \end{aligned}$$

Since $R(n,-1) = R(n,n+1) = 0$, we see that $h_{n+1} - f(n)h_n = \sum_{k=0}^{n} R(n,k)b_{k+1}$. Thus $g_n = \sum_{k=0}^{n} R(n,k)a_k = \sum_{k=0}^{n} R(n,k)(b_{k+1} - b_k) = h_{n+1} - (f(n)+1)h_n$ for $n \geqslant 0$. Since $h_0 = R(0,0)b_0$ and $h_n = g_n = 0$ for $n < 0$, Eq. (7) holds for $n < 0$ as well.

Eq. (7) defines a recurrence relation $h_{n+1} - (f(n)+1)h_n = g_n + R(0,0)b_0[n=-1]$, valid for all integer values of n. The solution to this recurrence is precisely Eq. (8). (See, for example, [10, pp. 46–48].) □

Theorems 2, 4, and 7 are all special cases of Theorem 9. Moreover, the difficulty in applying our approach to sums involving Stirling numbers of the second kind is perhaps more apparent now. The recurrence for the Stirling numbers of the second kind has f as a function of k, not n. If f is a function of k, f cannot be factored out of the summation after the first line in the proof of Theorem 9.

As an application of Theorem 9, we derive an expression for $\sum_{k=0}^{n} R_c(n,k)k^m$, where $R_c(n,k)$ satisfies the conditions in Theorem 9, $R_c(0,0)=1$, and $f(n)=c$ for all n.

Identity 47. $\sum_{k=0}^{n} R_c(n,k)k^{\underline{m}} = (c+1)^{n-m}n^{\underline{m}}$.

Proof. Theorem 9 immediately yields $\sum_{k=0}^{n} R_c(n,k) = (c+1)^n$. Fix $m \geqslant 1$, and suppose Identity 47 is true for $0, 1, 2, \ldots, m-1$. By Theorem 9, then, we have

$$\begin{aligned}\sum_{k=0}^{n} R_c(n,k)k^{\underline{m}} &= (c+1)^n m \sum_{k=0}^{n-1} \frac{(c+1)^{k-m+1}k^{\underline{m-1}}}{(c+1)^{k+1}} \\ &= (c+1)^{n-m}n^{\underline{m}}. \qquad \square\end{aligned}$$

Identity 47 and the relation $\binom{n}{j} j! = n^{\underline{j}}$ then imply the following generalization of Identity 12.

Identity 48. $\sum_{k=0}^{n} R_c(n,k)k^m = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \binom{n}{j} j!(c+1)^{n-j}$.

Interestingly enough, the power sum $\sum_{k=0}^{n} k^m$ cannot be evaluated via Identity 48 because the triangle of 1's that would need to be used in order to apply the identity does not satisfy, for all values of n and k, the recurrence relation for $R(n,k)$ in the hypotheses of Theorem 9.

7. Conclusions

Given sequences $\{a_k\}$ and $\{b_k\}$, with the relationship $\Delta b_k = a_k$, we have shown, for several different kinds of transforms, how the transform of $\{a_k\}$ relates to that of $\{b_k\}$. Unfortunately, our technique does not appear to work on sums involving Stirling numbers of the second kind, Eulerian numbers, or other numbers that do not satisfy the condition in Theorem 9 that f be a function only of n. Is there a way to modify our approach so that it applies to sums for which f is a function of k or of both n and k?

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